Transformation theory of the *n*-nonlinear equation

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Abstract

By using the *n*-chaotic function the *n*-nonlinear equation is transformed into the *n*-chaotic equation $dA/dt = \sum_{i=1}^{n} \frac{dN_i}{dt}$. If $dA/dt < 0$, it is stable. If $dA/dt > 0$, it is instable.

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1. Introduction

Since the discovery of the famous Lorenz chaotic system [1] over the past several decades chaos has become a very lively subject of scientific studies. The continuous efforts have been devoted to seeking the theory and methods of chaos. Many chaotic systems have been constructed in numerical simulations, but there are no analytic studies.

Consider the ordinary differenial equation

$$\frac{dA}{dt} = bA, \qquad (1)$$

where b is a constant.

$\frac{dA}{dt} = 0$ there is equilibrium.

The equation (1) has the solution

$$A = e^{bt} \qquad (2)$$

If $b < 0$, then $dA/dt < 0$. $A \to 0$ as $t \to \infty$. Its solution is convergent and stable. If $b > 0$, then $dA/dt > 0$. $A \to \infty$ as $t \to \infty$. Its solution is divergent and instable.

By using this method we study the *n*-nonlinear equaiton, Consider the *n*-nonlinear equation

$$\frac{dN_i}{dt} = F_i(N_1, \cdots, N_n) \qquad (3)$$

where $i = 1, \cdots, n.$

$\frac{dN_i}{dt}=0$ there are equilibria. By using the *n*-chaotic function [1,2] the *n*-nonlinear equation is transformed into the *n*-chaotic equation

$$\frac{dA}{dt}=\sum_{i=1}^{n}\frac{dN_i}{dt} \qquad (4)$$

If ${dA}/{dt}<0$, its soluion is stable and convergent. If ${dA}/{dt}>0$, its solution is instable and divergent.

2. Transformaion theory of the 3-nonlinear equation

In order to study the solutions of the 3-nonlinear equation we define the 3-chaotic function [2, 3]

$$\begin{aligned} N_1 &= \frac{1}{3}[A+2B\cos\theta], \\ N_2 &= \frac{1}{3}\left[A-2B\cos\left(\theta-\frac{\pi}{3}\right)\right], \\ N_3 &= \frac{1}{3}\left[A+2B\cos\left(\theta-\frac{2\pi}{3}\right)\right]. \end{aligned} \qquad (5)$$

The equation（5）is written into the matrix form

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}=\frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} A \\ 2B\cos\theta \\ 2B\sin\theta \end{pmatrix}. \qquad (6)$$

The equatoin（6）has the inverse transformation

$$\begin{pmatrix} A \\ B\cos\theta \\ B\sin\theta \end{pmatrix}=\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}. \qquad (7)$$

From（7）we obtain

$$\begin{aligned} A &= N_1+N_2+N_3, \\ B\cos\theta &= N_1-\frac{N_2+N_3}{2}, \\ B\sin\theta &= \frac{\sqrt{3}}{2}-\frac{N_2-N_3}{2}. \end{aligned} \qquad (8)$$

The equations （5）—（7）are the 3-nonlinear transformation group. From （8）we obtain the 3-chaotic equation

$$\frac{dA}{dt}=\frac{dN_1}{dt}+\frac{dN_2}{dt}+\frac{dN_3}{dt} \qquad （9）$$

This method allows us to study the 3-nonlinear equations including the Lorenz equation which produces the chaotic manifolds and nonlinear dynamics [4]

We study the 3-nonliear equation [2, 3, 5]

$$\begin{aligned}\frac{dN_1}{dt}&=N_1(1-N_1-a_1N_2-a_2N_3)\\ \frac{dN_2}{dt}&=N_2(1-a_2N_1-N_2-a_1N_3)\\ \frac{dN_3}{dt}&=N_3(1-a_1N_1-a_2N_2-N_3)\end{aligned} \qquad （10）$$

where $N_i(t)$ is the number of individuals of the $i-$th species and a_1 and a_2 denote interaction coefficients.

Equilibria: (0, 0, 0); 3 single-population solutions of the form (1, 0, 0); 3 two-population solutions of the form $(1-a_1,1-a_2,0)/(1-a_1a_2)$ and $(1,1,1)/\lambda_0$ where $\lambda_0=1+a_1+a_2$.

Substituting (10) into (9) we obtain the 3-chaotic equation

$$\frac{dA}{dt}=N_1+N_2+N_3-N_1^2-N_2^2-N_3^2-(a_1+a_2)(N_1N_2+N_1N_3+N_2N_3) \qquad （11）$$

Substituting (5) into (11) we obtain the 3-chaotic equation

$$\frac{dA}{dt}=A-\frac{1}{3}(\lambda_0A^2+2\lambda_1B^2) \qquad （12）$$

where $\lambda_1=1-\frac{a_1+a_2}{2}$.

We now discuss the stability and instability of the nonlinear terms of $\frac{dA}{dt}$.

$$\left(\frac{dA}{dt}\right)_{A^2}=-\frac{\lambda_0}{3}A^2, \qquad （13）$$

For $\lambda_0>0$, it is stable.

$$\left(\frac{dA}{dt}\right)_{B^2}=-\frac{2}{3}\lambda_1B^2. \qquad （14）$$

A necessary and sufficient condition for stability of the equation (14) is $\lambda_1>0$, namely $a_1+a_2<2$. If $\lambda_1<0$, namely $a_1+a_2>2$, it is instable.

Studying the stability and instability of the 3-nonlinear equation is transformed into studying the stability and instability of the 3-chaotic equation (9).

If $a_1 = a_2 = 1$, then $\lambda_1 = 3$ and $\lambda_1 = 0$. From (12) we obtain $A(t) = \frac{A(0)}{A(0) + (1 - A(0))e^{-t}}$.

It is stable.

We study the local solutions of equation (10). The equation (10) may be written into

$$\begin{aligned} \frac{d \ln N_1}{dt} &= 1 - N_1 - a_1 N_2 - a_2 N_3, \\ \frac{d \ln N_2}{dt} &= 1 - a_2 N_1 - N_2 - a_1 N_3, \\ \frac{d \ln N_3}{dt} &= 1 - a_1 N_1 - a_2 N_2 - N_3. \end{aligned} \qquad （15）$$

Setting $\lambda_0 N_i = e^{y_i}$, from the equation (15) we obtain

$$\begin{aligned} \frac{dy_1}{dt} &= 1 - \frac{1}{\lambda_0}(e^{y_1} + a_1 e^{y_2} + a_2 e^{y_3}), \\ \frac{dy_2}{dt} &= 1 - \frac{1}{\lambda_0}(a_2 e^{y_1} + e^{y_2} + a_1 e^{y_3}), \\ \frac{dy_3}{dt} &= 1 - \frac{1}{\lambda_0}(a_1 e^{y_1} + a_2 e^{y_2} + e^{y_3}). \end{aligned} \qquad （16）$$

We study the linear solutions of the equation (16). Setting $e^{y_i} = 1 + y_i$, from the equation (16) we obtain

$$\begin{aligned} \frac{dy_1}{dt} &= -\frac{1}{\lambda_0}(y_1 + a_1 y_2 + a_2 y_3), \\ \frac{dy_2}{dt} &= -\frac{1}{\lambda_0}(a_2 y_1 + y_2 + a_1 y_3). \\ \frac{dy_3}{dt} &= -\frac{1}{\lambda_0}(a_1 y_1 + a_2 y_2 + y_3), \end{aligned} \qquad （17）$$

The equation (17) has the following exact solutions

$$\begin{aligned} y_1 &= \left[e^{-t} + 2e^{\frac{-\lambda_1}{\lambda_0}t} \cos\left(\frac{\eta_1}{\lambda_0} t \right) \right], \\ y_2 &= \left[e^{-t} - 2e^{\frac{-\lambda_1}{\lambda_0}t} \cos\left(\frac{\eta_1}{\lambda_0} t - \frac{\pi}{3} \right) \right], \\ y_3 &= \left[e^{-t} + 2e^{\frac{-\lambda_1}{\lambda_0}t} \cos\left(\frac{\eta_1}{\lambda_0} t - \frac{2\pi}{3} \right) \right]. \end{aligned} \qquad （18）$$

where $\lambda_1 = 1 - \frac{(a_1 + a_2)}{2}$, $\eta_1 = \frac{\sqrt{3}}{2}(a_1 - a_2)$.

If $\lambda_1 > 0$, namely $a_1 + a_2 < 2$, it is stable. If $\lambda_1 < 0$, namely $a_1 + a_2 > 2$, it is instable. The results obtained from the equations (18) and (14) are the same.

We consider the Lorenz system [1, 4]

$$\begin{aligned} \frac{dN_1}{dt} &= \sigma(N_1 - N_2), \\ \frac{dN_2}{dt} &= \rho N_1 - N_2 - N_1 N_3, \\ \frac{dN_3}{dt} &= -\beta N_1 + N_1 N_2. \end{aligned} \tag{19}$$

It contains three constants: σ (the Prandtl number) ρ (the Rayleigh number), and β (an aspect ratio).

For $\rho > 1$, there are two equilibria:

$$(N_1, N_2, N_3) = (\pm\sqrt{\beta(\rho-1)}, \quad \pm\sqrt{\beta(\rho-1)}, (\rho-1)). \tag{20}$$

For $\rho < 1$, there is one equilibrium:

$$(N_1, N_2, N_3) = (0,0,0). \tag{21}$$

From (5), (9) and (19) we obtain the 3-chaotic equation

$$\frac{dA}{dt} = \frac{1}{3}[(\rho - 1 - \beta)A + (3\sigma + 2\rho - 2\beta + 1)B\cos\theta$$

$$+\sqrt{3}(\sigma+1)\sin\theta] - \frac{(2\sqrt{3})}{9}[AB\sin\theta + B^2 \sin 2\theta], \tag{22}$$

If $\frac{dA}{dt} < 0$, it is stable. If $\frac{dA}{dt} > 0$, it is instable.

We consider RÖssler system [4]. It has the following form:

$$\begin{aligned} \frac{dN_1}{dt} &= -N_2 - N_3, \\ \frac{dN_2}{dt} &= N_1 + aN_2 \\ \frac{dN_3}{dt} &= b + N_1 N_3 - cN_3 \end{aligned} \tag{23}$$

The system has two equilibria:

$$((c - \sqrt{c^2 - 4ab})/2, \quad (-c + \sqrt{c^2 - 4ab})/2a, \quad (c - \sqrt{c^2 - 4ab})/2a) \text{ and}$$

$$((c + \sqrt{c^2 - 4ab})/2, \quad -(c + \sqrt{c^2 - 4ab})/2a, \quad (c + \sqrt{c^2 - 4ab})/2a)$$

From (5), (9) and (23) we obtain the 3-chaotic equation

$$\frac{dA}{dt} = \frac{1}{3}[(a - c - 1)A + (4 - a + c)B\cos\theta - (a + c)\sqrt{3}B\sin\theta] + \frac{1}{9}[A^2 -$$

$$AB\cos\theta+\sqrt{3}AB\sin\theta+2\sqrt{3}B^2\sin\theta-2B^2\cos^2\theta]$$

If $\frac{dA}{dt}<0$, it is stable. If $\frac{dA}{dt}>0$, it is instable.

3. Transformation theory of the 4-nonlinear equation

In order to study the solutions of the 4-nonliear equation we define the 4-chaotic function

$$\begin{aligned}
N_1&=\frac{1}{4}[A_1+A_2+2H\cos\beta],\\
N_2&=\frac{1}{4}[A_1-A_2-2H\sin\beta],\\
N_3&=\frac{1}{4}[A_1+A_2-2H\cos\beta],\\
N_4&=\frac{1}{4}[A_1-A_2+2H\sin\beta]
\end{aligned}\tag{24}$$

The equation (24) is written into matrix form

$$\begin{pmatrix}N_1\\N_2\\N_3\\N_4\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1&1&1&0\\1&-1&0&-1\\1&1&-1&0\\1&-1&0&1\end{pmatrix}\begin{pmatrix}A_1\\A_2\\2H\cos\beta\\2H\sin\beta\end{pmatrix}\tag{25}$$

The equatoin (25) has the inverse transformation

$$\begin{pmatrix}A_1\\A_2\\H\cos\theta\\H\sin\theta\end{pmatrix}=\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&0&-1&0\\0&-1&0&1\end{pmatrix}\begin{pmatrix}N_1\\N_2\\N_3\\N_4\end{pmatrix}\tag{26}$$

From (26) we obtain

$$\begin{aligned}
A_1&=N_1+N_2+N_3+N_4,\\
A_2&=N_1-N_2+N_3-N_4,\\
H\cos\beta&=N_1-N_3,
\end{aligned}\tag{27}$$

From (27) we obtain the 4-chaotic equation

$$\frac{dA_1}{dt}=\frac{dN_1}{dt}+\frac{dN_2}{dt}+\frac{dN_3}{dt}+\frac{dN_4}{dt}\tag{28}$$

We consider the 4-nonliear equation

$$\begin{aligned}
\frac{dN_1}{dt}&=N_1(1-N_1-a_1N_2-a_2N_3-a_3N_4),\\
\frac{dN_2}{dt}&=N_2(1-a_3N_1-N_2-a_1N_3-a_2N_4),
\end{aligned}\tag{29}$$

$$\frac{dN_3}{dt} = N_3(1 - a_2N_1 - a_3N_2 - N_3 - a_1N_4),$$
$$\frac{dN_4}{dt} = N_4(1 - a_1N_1 - a_2N_2 - a_3N_3 - N_4),$$

where N_i is the number of individuals of the $i-$th species. a_1, a_2 and a_3 denote interaction coefficiens.

Equilibria: (0, 0, 0, 0); 4 single-population solutions of the form (1, 0, 0, 0); 6 two-population solution of the form $(1-a_1, 1-a_3, 0, 0)/(1-a_1a_3)$; 4 three-population solutions of the form $D_1, D_2, D_3, 0)/D_4$, where $D_1 = 1 + a_1^2 + a_2a_3 - a_1 - a_2 - a_1a_3$, $D_2 = 1 + a_1a_2 + a_2a_3$, $-a_1 - a_3 - a_2^2$, $D_3 = 1 + a_1a_2 + a_3^2 - a_2 - a_3 - a_1a_3$, $D_4 = 1 + a_1^2a_2 + a_2a_3^2 - a_2^2 - 2a_1a_3$, and $(1,1,1,1)/(1 + a_1 + a_2 + a_3)$.

By substituting (24) and (29) into (28) we obtain the 4-chaotic equation

$$\frac{dA_1}{dt} = A_1 - \frac{1}{4}(\lambda_{A_1}A_1^2 + \lambda_{A_2}A_2^2 + 2\lambda_H H^2), \qquad (30)$$

where $\lambda_{A_1} = 1 + a_1 + a_2 + a_3$, $\lambda_{A_2} = 1 - a_1 + a_2 - a_3$, $\lambda_H = 1 - a_2$.

We will discuss the stability and instability of the nonlinear terms of dA_1/dt in the equation (30)

$$\left(\frac{dA_1}{dt}\right)_{A_1^2} = A_1 - \frac{1}{4}\lambda_{A_1}A_1^2. \qquad (31)$$

For $\lambda_{A_1} > 0$, it is stable.

$a_1, \cdots, a_4$ are interaction coefficients. $\left(\frac{dA_1}{dt}\right)_{A_2^2} = -\frac{1}{4}\lambda_{A_2}A_2^2$.

（32）

If $\lambda_{A_2} > 0$, it is stable. If $\lambda_{A_2} < 0$, it is unstable.

$$\left(\frac{dA_1}{dt}\right)_{H^2} = -\frac{\lambda_H H^2}{2}. \qquad (33)$$

If $\lambda_H > 0$, it is stable; if $\lambda_H < 0$, it is instable.

If $a_1 = a_2 = a_3 = a_4 = 1$, then $\lambda_{A_1} = 4$, $\lambda_{A_2} = \lambda_H = 0$. From (31) we obtain

$$A_1(t) = \frac{A_1(0)}{A_1(0) + [1 - A_1(0)]e^{-t}}.$$

It is stable.

Now we study the local solutions of the equation (29). The equation (29) is written into

$$\begin{aligned}
\frac{d\ln N_1}{dt} &= 1 - N_1 - a_1N_2 - a_2N_3 - a_3N_4, \\
\frac{d\ln N_2}{dt} &= 1 - a_3N_1 - N_2 - a_1N_3 - a_2N_4, \\
\frac{d\ln N_3}{dt} &= 1 - a_2N_1 - a_3N_2 - N_3 - a_1N_4, \\
\frac{d\ln N_4}{dt} &= 1 - a_1N_1 - a_2N_2 - a_3N_3 - N_4,
\end{aligned} \tag{34}$$

Setting $\lambda_{A_1} N_j = e^{y_j}$, the equation (34) is transformed into

$$\begin{aligned}
\frac{dy_1}{dt} &= 1 - \frac{1}{\lambda_{A_1}}(e^{y_1} + a_1e^{y_2} + a_2e^{y_3} + a_3e^{y_4}),. \\
\frac{dy_2}{dt} &= 1 - \frac{1}{\lambda_{A_1}}(a_3e^{y_1} + e^{y_2} + a_1e^{y_3} + a_2e^{y_4}), \\
\frac{dy_3}{dt} &= 1 - \frac{1}{\lambda_{A_1}}(a_2e^{y_1} + a_3e^{y_2} + e^{y_3} + a_1e^{y_4}), \\
\frac{dy_4}{dt} &= 1 - \frac{1}{\lambda_{A_1}}(a_1e^{y_1} + a_2e^{y_2} + a_3e^{y_3} + e^{y_4}),
\end{aligned} \tag{35}$$

Setting $e^{y_i} = 1 + y_i$, from the equation (35) we obtain

$$\begin{aligned}
\frac{dy_1}{dt} &= -\frac{1}{\lambda_{A_1}}(y_1 + a_1y_2 + a_2y_3 + a_3y_4), \\
\frac{dy_2}{dt} &= -\frac{1}{\lambda_{A_1}}(a_3y_1 + y_2 + a_1y_3 + a_2y_4), \\
\frac{dy_3}{dt} &= -\frac{1}{\lambda_{A_1}}(a_2y_1 + a_3y_2 + y_3 + a_1y_4), \\
\frac{dy_4}{dt} &= -\frac{1}{\lambda_{A_1}}(a_1y_1 + a_2y_2 + a_3y_3 + y_4).
\end{aligned} \tag{36}$$

The equation (36) has the following exact solutions

$$y_1 = e^{-t} + 2\exp\left(-\frac{\lambda_H t}{\lambda_{A_1}} t\right)\cos\left(\frac{\lambda_\beta t}{\lambda_{A_1}}\right) + \exp\left(-\frac{\lambda_{A_2}}{\lambda_{A_1}} t\right),$$

$$y_2 = e^{-t} - 2\exp\left(-\frac{\lambda_H t}{\lambda_{A_1}} t \right)\sin\left(\frac{\lambda_\beta t}{\lambda_{A_1}} \right) - \exp\left(-\frac{\lambda_{A_2}}{\lambda_{A_1}} t \right), \qquad (37)$$

$$y_3 = e^{-t} - 2\exp\left(-\frac{\lambda_H t}{\lambda_{A_1}} t \right)\cos\left(\frac{\lambda_\beta t}{\lambda_{A_1}} t \right) + \exp\left(-\frac{\lambda_{A_2}}{\lambda_{A_1}} t \right),$$

$$y_4 = e^{-t} + 2\exp\left(-\frac{\lambda_H t}{\lambda_{A_1}} t \right)\sin\left(\frac{\lambda_\beta t}{\lambda_{A_1}} t \right) - \exp\left(-\frac{\lambda_{A_2}}{\lambda_{A_1}} t \right).$$

where $\lambda_\beta = -a_1 + a_3$.

A necessary and sufficient condition for stability of the equation (37) is $\lambda_H > 0$ and $\lambda_{A_2} > 0$. If $\lambda_H < 0$ and $\lambda_{A_2} > 0$, then the solutions are instable. The results obtained from the equations (30) and (37) are the same.

4. Transformation theory of the 5-nonliear equation

In order to study the solutions of the 5-nonliear equation we define the 5-chaotic function

$$N_i = \frac{1}{5}\left[A + 2\sum_{j=1}^{2} (-1)^{(i-1)j} B_j \cos(\theta_j + (-1)^j \frac{(i-1)j\pi}{5}) \right]. \qquad (38)$$

From (38) we obtain

$$A = \sum_{i=1}^{5} N_i \qquad (39)$$

From (39) we obtain the 5-chaotic equation

$$\frac{dA}{dt} = \sum_{i=1}^{5} \frac{dN_i}{dt} \qquad (40)$$

we consider the 5-nonlinear equation

$$\frac{dN_i}{dt} = N_i \left(1 - \sum_{i=1}^{5} a_{ij} N_j \right) \qquad (41)$$

where

$$(a_{ij}) = \begin{pmatrix} 1 & a_4 & a_3 & a_2 & a_1 \\ a_1 & 1 & a_4 & a_3 & a_2 \\ a_2 & a_1 & 1 & a_4 & a_3 \\ a_3 & a_2 & a_1 & 1 & a_4 \\ a_4 & a_3 & a_2 & a_1 & 1 \end{pmatrix}$$

N_i is the number of individuals of the *i*-th species. $a_1, \cdots, a_4$ are interaction coefficients.

Equilibria: $(0,0,0,0,0)$; $\cdots$; $(1,1,1,1,1)/\lambda_0$, where $\lambda_0 = 1 + a_1 + a_2 + a_3 + a_4$.

Substituting (39) and (41) into (40) we obtain the 5-chaotic equation

$$\frac{dA}{dt} = A - \frac{1}{5}[\lambda_0 A^2 + 2\lambda_1 B_1^2 + 2\lambda_1 B_2^2] \qquad (42)$$

where $\lambda_1 = 1 - (a_1 + a_4) + \cos\frac{\pi}{5} + (a_2 + a_3)\cos\frac{2\pi}{5}$,

$\lambda_2 = 1 + (a_1 + a_4) + \cos\frac{2\pi}{5} + (a_2 + a_3)\cos\frac{5\pi}{5}$. $\frac{dA}{dt} = 1 - \frac{\lambda_0 A^2}{5}$ is stable. If $\lambda_1 > 0$ and $\lambda_2 > 0$, it is stable. If $\lambda_1 < 0$ and $\lambda_2 > 0$, $\lambda_1 > 0$ and $\lambda_2 < 0$, it is instable.

If $a_1 = a_2 = a_3 = a_4 = 1$, then $\lambda_0 = 5$, $\lambda_1 = 0$ and. $\lambda_2 = 0$. From (42) we obtain

$A(t) = \frac{A(0)}{A(0) + [1 - A(0)]e^{-t}}$. It is stable.

5. Transformation theory of the *n*-nonliear equation

We suggest a new method for studying the *n*-nonlinear equation that studying the stability and instability of the *n*-nonlinear equation is transformed into studying stability and instability of the *n*-chaotic equation [2]

$$\frac{dA}{dt} = \sum_{i=1}^{n} \frac{dN_i}{dt}. \qquad (43)$$

If $\frac{dA}{dt} < 0$, it is stable. If $\frac{dA}{dt} > 0$, it is instable.

This method allows us to study the *n*-nonlinear equation via the *n*-chaotic function. This process is illustrated by the diagram of Fig.1.

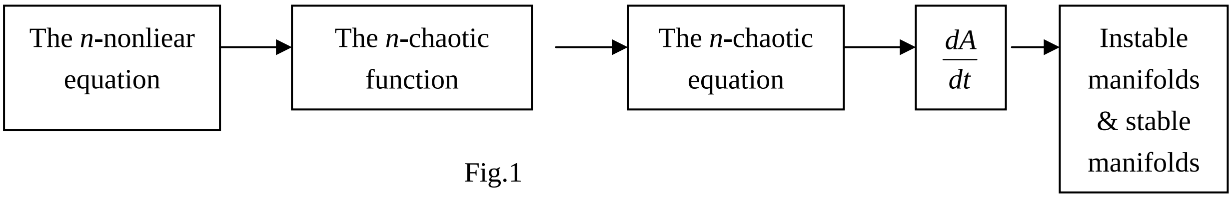

Fig.1

We provide a new mathematical tool for studying the complex systems and network science.

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